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Minkowski type inequality for fuzzy and pseudo-integrals

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Abstract

One of the famous mathematical inequality is Minkowski's inequality. It is an important inequality from both mathematical and application points of view. In this paper, a Minkowski type inequality for fuzzy and pseudo-integrals is studied. The established results are based on the classical Minkowski's inequality for integrals.

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1 Introduction

The theory of fuzzy measures and fuzzy integrals was introduced by Sugeno [37] as a tool for modelling non-determinstic problems. Fuzzy integrals or Sugeno integrals have very interesting properties from a mathematical point of view which have been studied by many authors, studied by many authors including Pap [24], Ralescu and Adams [26], Wang and Klir [40] among others. Ralesco and Adams [26] studied several equivalent definitions of fuzzy integrals, while Pap [24] and Wang and Klir [40], provided an overview of fuzzy measure theory. The fuzzy integral for monotone functions was presented in [27]. In fact, fuzzy measures and fuzzy integrals are versatile operators which can be used in different areas. They have a broad use in information fusion, electronic auctions, decision making, and etc. Chen et al. [4] employed fuzzy integral and fuzzy measure to establish a public attitude analysis model. The integral inequalities are useful results in several theoretical and applied fields. For instance, integral inequalities play a major role in the development of a time scales calculus. $\ddot{O}$zkan et al. [22] obtained H$\ddot{o}$lders inequality, Minkowskis inequality and Jensen's inequality on time scales. Also H. M. Srivastava et.al [34, 35] studied some generalizations of Maroni's inequality and some weighted Pial-type inequalities on time scales. Some famous inequalities have been generalized to fuzzy integral. Román-Flores and Chalco-Cano [28] analyzed an interesting type of geometric inequalities for fuzzy integral with some applications to convex geometry. Román-Flores et al. [29, 30] studied a Jensen type inequality and a convolution type inequality for fuzzy integrals. Also, they have investigated a Chebyshev type inequality and a Stolarsky type inequality for fuzzy integrals [12, 31]. In [12], a fuzzy Chebyshev inequality for a special case was obtained which has been generalized by Ouyang et al. [21]. Furthermore, Chybyshev type inequalities for fuzzy integral were proposed in a rather general form by Mesiar and Ouyang [17]. Recently, B. Daraby and L. Arabi Proved a related Fritz Carlson type inequality for Sugeno integrals [8]. For more refrences on integral inequalities and its applications you can see [39, 20, 41, 36].

Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers a semiring is taken on a real interval $[a, b] \subset [-\infty, \infty]$ endowed with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$ ([11, 9, 8, 5, 25, 6, 37]). Based on this structure there where developed

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the concepts of $\oplus$-measure (pseudo-additive measure), pseudo-integral, pseudo-convolution, pseudo-Laplace transform and etc. ([7, 12, 27]).

This paper is organized as follows: In Section 2 some preliminaries and summarization of some previous known results are given. Section 3 proposes a Minkowski type inequality for fuzzy integrals. Section 4, deals with a Minkowski type inequality for Pseudo-integrals. Finally, Section 5 contains a short conclusion.

2 Preliminaries

In this section, some definitions and basic properties of the Sugeno and Pseudo integrals which will be used in the next sections are presented.

Definition 2.1. Let Σ be a σ-algebra of subsets of X and let $\mu : \Sigma \to [0, \infty)$ be a non-negative, extended real-valued set function, we say that μ is a fuzzy measure iff:

(FM1) $\mu(\varnothing) = 0$;

(FM2) $E, F \in \sum$ and $E \subseteq F$ imply $\mu(E) \leq \mu(F)$ (monotonicity);

(FM3) $E_n \subseteq \sum$, $\quad E_1 \subseteq E_2 \subseteq \dots$ imply $\quad \lim \mu(E_n) = \mu(\bigcup_{i=1}^{\infty} E_n)$ (continuity from below);

(FM4) $E_n \subseteq \sum$, $\quad E_1 \supseteq E_2 \supseteq \dots, \mu(E_1) < \infty$ imply $\lim \mu(E_n) = \mu(\bigcap_{i=1}^{\infty} E_n)$ (continuity from above).

If f is a non-negative real-valued function on X, we will denote $F_\alpha = \{x \in X \mid f(x) \geq \alpha\} = \{f \geq \alpha\}$, the $\alpha - level$ of f, for $\alpha > 0$. $F_0 = \{x \in X \mid f(x) > 0\} = supp(f)$ is the support of f. We know that: $\alpha \leq \beta \Rightarrow \{f \geq \beta\} \subseteq \{f \geq \alpha\}$.

If μ is a fuzzy measure on X , we define the following:

$$\mathfrak{F}^\mu(X) = \{f : X \to [0, \infty)| \quad \text{f is } \mu - \text{measurable}\}.$$

Definition 2.2. Let μ be a fuzzy measure on (X, Σ). If $f \in \mathfrak{F}^\mu(X)$ and $A \in \Sigma$, then the Sugeno integral (or fuzzy integral) of f on A, with respect to the fuzzy measure μ, is defined [40] as

$$⨍_A f d\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(A \cap F_\alpha)).$$

Where $\vee$, $\wedge$ denotes the operation sup and inf on $[0, \infty)$ respectively. In particular, if $A = X$ then:

$$⨍_X f d\mu = ⨍ f d\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(F_\alpha)).$$

The following proposition gives most elementary properties of the fuzzy integral and can be found in [40].

Proposition 2.3. Let $(X, \mathfrak{F}, \mu)$ be a fuzzy measure space, with $A, B \in \sum$ and $f, g \in \mathfrak{F}$. We have

1. $⨍_A f d\mu \leq \mu(A)$.
2. $⨍_A k d\mu \leq k \wedge \mu(A)$, for k nonnegative constant.

3. If $f \leq g$ on A, then $⨍_A f d\mu \leq ⨍_A g d\mu$.
4. if $A \subset B$, then $⨍_A f d\mu \leq ⨍_A g d\mu$.
5. if $\mu(A) < \infty$, then $⨍_A f d\mu \geq \alpha \Leftrightarrow \mu(A \cap \{f \geq \alpha\}) \geq \alpha$.
6. $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \Rightarrow ⨍_A f d\mu \leq \alpha$.
7. $⨍_A f d\mu < \alpha \Leftrightarrow$ there exists $\gamma < \alpha$ such that $(A \cap \{f \geq \gamma\}) < \alpha$.
8. $⨍_A f d\mu > \alpha \Leftrightarrow$ there exists $\gamma > \alpha$ such that $(A \cap \{f \geq \gamma\}) > \alpha$.

Remark 2.4. Let $F(\alpha) = \mu(A \cap F_\alpha)$, from parts (5) and (6) of the above Proposition, it very important to note that

$$F(\alpha) = \alpha \Rightarrow ⨍_A f d\mu = \alpha.$$

Thus, from a numerical point of view, the Sugeno integral can be calculated by solving the equation $F(\alpha) = \alpha$.

Let $[a, b]$ be a closed (in some cases can be considered semiclosed) subinterval of $[-\infty, \infty]$. The full order on $[a, b]$ will be denoted by $\preceq$.

The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a, b] \times [a, b] \to [a, b]$ which is commutative, nondecreasing (with respect to $\preceq$), associative and with a zero (neutral) element denoted by $\mathbf{0}$, i.e., for each $x \in [a, b], \mathbf{0} \oplus x = x$ holds (usually $\mathbf{0}$ is either a or b). Let $[a, b]_+ = \{x | x \in [a, b], \mathbf{0} \preceq x\}$.

Definition 2.5. The operation $\odot$ (pseudo-multiplication) is a function $\odot : [a, b] \times [a, b] \to [a, b]$ which is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a, b]_+$, associative and for which there exists a unit element $\mathbf{1} \in [a, b]$, i.e., for each $x \in [a, b], \mathbf{1} \odot x = x$.

We assume also $\mathbf{0} \odot x = \mathbf{0}$ that $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e., $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$. The structure $([a, b], \oplus, \odot)$ is a semiring ([14]). In this paper, we will consider semirings with the following continuous operations:

Case I: The pseudo-addition is idempotent operation and the pseudo-multiplication is not.
(a) $x \oplus y = \sup(x, y)$, $\odot$ is arbitrary not idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0} = a$ and the idempotent operation sup induces a full order in the following way: $x \preceq y$ if and only if $\sup(x, y) = y$.
(b) $x \oplus y = \inf(x, y)$, $\odot$ is arbitrary not idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0}= b$ and the idempotent operation *inf* induces a full order in the following way: $x \preceq y$ if and only if $\inf(x, y) = y$.

Case II: The pseudo-operations are defined by a monotone and continuous function $g : [a, b] \to [0, \infty]$, i.e., pseudo operations are given with $x \oplus y = g^{-1}(g(x) + g(x))$ and $x \odot y = g^{-1}(g(x)g(x))$. If the zero element for the pseudo-addition is a, we will consider increasing generators. Then $g(a) = 0$ and $g(b) = \infty$. If the zero element for the pseudo-addition is b, we will consider decreasing generators. Then $g(b) = 0$ and $g(a) = \infty$. If the generator g is increasing (respectively decreasing), then the operation $\oplus$ induces the usual order (respectively opposite to the usual order) on the interval $[a, b]$ in the following way: $x \preceq y$ if and only if $g(x) \leq g(y)$.

Case III: Both operations are idempotent. We have
(a) $x \oplus y = \sup(x, y), x \odot y = \inf(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = a$ and $\mathbf{1} = b$. The

idempotent operation *sup* induces the usual order ($x \preceq y$ if and only if $\sup(x, y) = y$).
(b) $x \oplus y = \inf(x, y), x \odot y = \sup(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = b$ and $\mathbf{1} = a$. The idempotent operation *inf* induces an order opposite to the usual order ($x \preceq y$ if and only if $\inf(x, y) = y$).

Let X be a non-empty set. Let $\mathbb{A}$ be a σ-algebra of subsets of a set X.

We shall consider the semiring $([a, b], \oplus, \odot)$, when pseudo-operations are generated by a monotone and continuous function $g : [a, b] \to [0, \infty]$, i.e., pseudo-operations are given with $x \oplus y = g^{-1}(g(x) + g(y))$ and $x \odot y = g^{-1}(g(x)g(y))$.

Then the pseudo-integral for a function $f : [c, d] \to [a, b]$ reduces on the g-integral

$$\int_{[c,d]}^{\oplus} f(x)dx = g^{-1}\Big(\int_{c}^{d} g(f(x))dx\Big). \tag{2.1}$$

More on this structure as well as corresponding measures and integrals can be found in ([23]). The second class is when $x \oplus y = \max(x, y)$ and $x \odot y = g^{-1}(g(x)g(y))$, the pseudo-integral for a function $f : \mathbb{R} \to [a, b]$ is given by

$$\int_{\mathbb{R}}^{\oplus} f \odot dm = \sup\Big(f(x) \odot \psi(x)\Big),$$

where function ψ defines sup-measure m. Any sup-measure generated as essential supremum of a continuouse denisty can be obtained as a limit of pseudo-additive measures with respect to generated pseudo-additive. For any continuouse function $f : [0, \infty] \to [0, \infty]$ the integral $\int^{\oplus} f \odot dm$ can be obtained as a limit of g-integrals. We denoted by μ the usual Lebesgue measure on $\mathbb{R}$. We have

$$m(A) = ess \sup(x|x \in A) = \sup\{a|\mu(x|x \in A, x > a) > 0\}.$$

Theorem 2.6. ([18]). Let m be a sup-measure on $([0, \infty], \mathbb{B}[0, \infty])$, where $\mathbb{B}([0, \infty])$ is the Borel σ-algebra on $[0, \infty]$, $m(A) = ess \sup_{\mu}(\psi(x)|x \in A)$, and $\psi : [0, \infty] \to [0, \infty]$ is a continuouse density. Then for any pseudo-addition $\oplus$ with a generator g there exists a family m_λ of $\oplus_\lambda$-measure on $([0, \infty], \mathbb{B})$, where $\oplus_\lambda$ is a generated by g^λ (the function g of the power λ), $\lambda \in (0, \infty)$, such that $\lim_{\lambda \to \infty} m_\lambda = m$.

Theorem 2.7. ([18]). Let $([0, \infty], \sup, \odot)$ be a semiring, when $\odot$ is a generated with g, i.e., we have $x \odot y = g^{-1}(g(x)g(y))$ for every $x, y \in (0, \infty)$. Let m be the same as in Theorem 2.6, Then there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$ -measures, where $\oplus_\lambda$ is a generated by $g^\lambda, \lambda \in (0, \infty)$ such that for every continuous function $f : [0, \infty] \to [0, \infty]$,

$$\int^{\sup} f \odot dm = \lim_{\lambda \to \infty} \int^{\oplus_\lambda} f \odot dm_\lambda = \lim_{\lambda \to \infty} (g^\lambda)^{-1}\Big(\int g^\lambda(f(x))dx\Big).$$

Easyly a stright calculus give the following Lemma:

Lemma 2.8. Let f_1 and f_2 be integrable functions, $A \in \sum$ and $f_1 \leq f_1$, so we have:

1. $⨍_A f_1 dx \leq ⨍_A f_2 dx$.
2. $\int_A^{\oplus} f_1 dx \leq \int_A^{\oplus} f_2 dx$.

The classical Minkowski's inequality was published by Minkowski [19] in his famous book "Geometrie der Zahlen". A proof of Minkowski's inequality as well as several extensions, related results, and interesting geometrical interpretations can be found in [32, 33]. An extension of Minkowski's inequality, which is based on Hölder's inequality, is given in [40]. Applications of Minkowski's inequality have been studied by many authors. For example özkan et al. [22] applied Minkowski's inequality, Hölder's inequality and Jensen's inequality on time scales. Lu et al. [15] used Minkowski's inequality for fast full search in motion estimation. The classical Minkowski's inequality [19] is as follows:

$$\left(\int_a^b (f(x)+g(x))^s dx\right)^{\frac{1}{s}} \leq \left(\int_a^b f(x)^s dx\right)^{\frac{1}{s}} + \left(\int_a^b g(x)^s dx\right)^{\frac{1}{s}} \tag{2.2}$$

where $1 \leq s < \infty$ and $f, g : [0,1] \to [0,\infty)$ are two nonnegative functions.
Note we recall the following inequalities which are the fuzzy versions of Minkowski's inequality at two cases and appears in [1].

Theorem 2.9. Let $f, g : [0,1] \to [0,\infty)$ be two real valued functions and let μ be the Lebesgue measure on $\mathbb{R}$. If f, g are both continuous and strictly decreasing functions, then the inequality

$$\left(⨍_0^1 (f+g)^s d\mu\right)^{\frac{1}{s}} \leq \left(⨍_0^1 f^s d\mu\right)^{\frac{1}{s}} + \left(⨍_0^1 g^s d\mu\right)^{\frac{1}{s}}$$

holds for all $1 \leq s < \infty$.

Theorem 2.10. Let $f, g : [0,1] \to [0,\infty)$ be two real valued functions and let μ be the Lebesgue measure on $\mathbb{R}$. If f, g are both continuous and strictly increasing functions, then the inequality

$$\left(⨍_0^1 (f+g)^s d\mu\right)^{\frac{1}{s}} \leq \left(⨍_0^1 f^s d\mu\right)^{\frac{1}{s}} + \left(⨍_0^1 g^s d\mu\right)^{\frac{1}{s}} \tag{2.3}$$

holds for all $1 \leq s < \infty$.

The following theorem is pseudo version of Minkoeski's inequality and appears in [2].

Theorem 2.11. Let $f, g : X \to [0,\infty)$ be two measurable functions and $s \in [1,\infty)$. If an additive generator $g : [a,b] \to [0,1]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ are increasing. Then for any $\sigma - \oplus$-measure m it holds:

$$\left(\int_X^{\oplus} (f+g)^s d\mu\right)^{\frac{1}{s}} \leq \left(\int_X^{\oplus} f^s d\mu\right)^{\frac{1}{s}} + \left(\int_X^{\oplus} g^s d\mu\right)^{\frac{1}{s}} \tag{2.4}$$

The following theorem shows the new classical version of Minkowski's inequality and appears in [3].

Theorem 2.12. Let f and g be positive functions satisfying
$0 < m \leq \dfrac{f(x)}{g(x)} \leq M, \quad \forall x \in [a,b]$, we have

$$\left(\int_a^b f^s(x)dx\right)^{\frac{1}{s}} + \left(\int_a^b g^s(x)dx\right)^{\frac{1}{s}} \leq c\left(\int_a^b (f(x)+g(x))^s dx\right)^{\frac{1}{s}}, \tag{2.5}$$

where $1 \leq s < \infty$ and $c = \dfrac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

3 Minkowski's inequality for fuzzy integrals

In this section, by an example we show that the Theorem 2.12 is not valid for the Sugeno integral.

Example 3.1. Let $f(x) = x+1$, $g(x) = 2x+1$ and $s = 1$. We have $0 < \frac{1}{3} \leq \frac{f(x)}{g(x)} \leq 1$ and

(i) $$⨍_0^1 f(x)d\mu = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \mu(\{x+1 \geq \alpha\})] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (\alpha - 1)] = 1,$$

(ii) $$⨍_0^1 g(x)d\mu = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \mu(\{2x+1 \geq \alpha\})] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (\frac{\alpha-1}{2})] = 1,$$

(iii) $$⨍_0^1 (f(x)+g(x))d\mu = \bigvee_{\alpha\in[0,2]} [\alpha \wedge \mu(\{3x+2 \geq \alpha\})] = \bigvee_{\alpha\in[0,2]} [\alpha \wedge (\frac{\alpha-2}{3})] = \frac{5}{4},$$

Consequently,

$$2 = ⨍_0^1 f(x)d\mu + ⨍_0^1 g(x)d\mu \nleq c⨍_0^1 (f(x)+g(x))d\mu = \frac{10}{8} \times \frac{5}{4} = \frac{50}{32}.$$

inequality (2.5) is not valid for fuzzy integrals.

Tn the following theorem we show a Minkoeski tupe inequality derived from (2.5) for the Sugeno integral.

Theorem 3.2. (Fuzzy Minkowski's inequality, decreasing case). Let $f, g : [0,1] \to [0,\infty)$ be two real valued and non-negative functions and let μ be the Lebesgue measure on $\mathbb{R}$. Let f, g be both continuous and strictly decreasing functions. If functions satisfying

$$0 < m \leq \frac{f(x)}{g(x)} \leq M, \quad \forall x \in [0,1]$$

then the inequality

$$\Big(⨍_0^1 f^s(x)d\mu\Big)^{\frac{1}{s}} + \Big(⨍_0^1 g^s(x)d\mu\Big)^{\frac{1}{s}} \leq 2c\Big(⨍_0^1 (f(x)+g(x))^s d\mu\Big)^{\frac{1}{s}}, \tag{3.1}$$

holds, where $1 \leq s < \infty$ and $c = \dfrac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

Proof. Since $\frac{f(x)}{g(x)} \leq M, f \leq M(f(x)+g(x)) - Mf(x)$. Therefore

$$(M+1)^s f(x)^s \leq M^s (f(x)+g(x))^s$$

and so,

$$f(x)^s \leq \frac{M^s}{(M+1)^s}(f(x)+g(x))^s.$$

Now we have

$$\Big(⨍_0^1 f(x)^s d\mu\Big)^{\frac{1}{s}} \leq \Big(⨍_0^1 (\frac{M}{M+1})^s (f(x)+g(x))^s d\mu\Big)^{\frac{1}{s}}. \tag{3.2}$$

By Lemma 2.8 (1) we have

$$⨍_0^1 \frac{M}{M+1} dx < ⨍_0^1 1dx = 1. \tag{3.3}$$

So by (3.2) and (3.3) we can write

$$\left(⨍_0^1 f(x)^s d\mu\right)^{\frac{1}{s}} \leq \left(⨍_0^1 (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}}. \tag{3.4}$$

On the other hand, since $mg(x) \leq f(x)$, Hence

$$g \leq \frac{1}{m}(f(x)+g(x)) - \frac{1}{m}g(x).$$

Therefore,

$$(\frac{1}{m}+1)^s g(x)^s \leq (\frac{1}{m})^s (f(x)+g(x))^s,$$

and so, by Lemma 2.8 (1) we have

$$\left(⨍_0^1 g(x)^s d\mu\right)^{\frac{1}{s}} \leq \left(⨍_0^1 (\frac{1}{m+1})^s (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}}. \tag{3.5}$$

Since $\dfrac{1}{m+1} < 1$, then

$$⨍_0^1 \frac{1}{m+1} dx < ⨍_0^1 1dx = 1. \tag{3.6}$$

The inequalities (3.5) and (3.6) follows that

$$\left(⨍_0^1 g(x)^s d\mu\right)^{\frac{1}{s}} \leq \left(⨍_0^1 (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}}. \tag{3.7}$$

Now with adding the inequalities (3.4) and (3.7):

$$\left(⨍_0^1 f(x)^s d\mu\right)^{\frac{1}{s}} + \left(⨍_0^1 g(x)^s d\mu\right)^{\frac{1}{s}} \le 2\left(⨍_0^1 (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}}$$
$$\le 2c\left(⨍_0^1 (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}}.$$

The proof is now complete. Q.E.D.

Example 3.3. Let $f, g : [0,1] \to [0,\infty)$ be two real valued functions defined as $f(x) = 1-x$, $g(x) = 1-x^2$ and μ be the Lebesgue measure on $\mathbb{R}$. Let $s = 1$. A straightforward calculus shows that $0 < \frac{1}{2} \le \frac{f}{g} \le 1$ and

(i) $$⨍_0^1 f(x)d\mu = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \mu(\{1-x \ge \alpha\})] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1-\alpha)] = \frac{1}{2} = 0.5,$$

(ii) $$⨍_0^1 g(x)d\mu = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \mu(\{1-x^2 \ge \alpha\})] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \sqrt{1-\alpha}] = 0.618,$$

(iii) $$⨍_0^1 (f+g)d\mu = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \mu(\{-x^2-x+2 \ge \alpha\})]$$
$$= \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \left(-\frac{1}{2}+\frac{1}{2}\sqrt{(9-4\alpha)}\right)\right] = 0.732.$$

Therefore

$$1.118 = 0.5 + 0.618 = \left(⨍_0^1 f d\mu\right) + \left(⨍_0^1 g d\mu\right) \le 2c\left(⨍_0^1 (f+g)d\mu\right)$$
$$= 2c \times 0.732$$
$$= 1.464c.$$

Theorem 3.4. (Fuzzy Minkowski's inequality, decreasing case). Let $f, g : [0,1] \to [0,\infty)$ be two real valued and non-negative functions and let μ be the Lebesgue measure on $\mathbb{R}$. Let f, g be both continuous and strictly decreasing functions and satisfying

$$0 < m \le \frac{f(x)}{g(x)} \le M, \quad \forall x \in [0,1]$$

then the inequality

$$\left(⨍_0^1 f(x)^s d\mu\right)^{\frac{1}{s}} + \left(⨍_0^1 g(x)^s d\mu\right)^{\frac{1}{s}} \le 2c\left(⨍_0^1 (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}},$$

holds, where $1 \le s < \infty, n \ge 2$ and $c = \dfrac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

Proof. The proof is similar Theorem 3.2. Q.E.D.

4 Minkowski's inequality for pseudo-integrals

Our purpose in this section is to prove the Minkowski's inequality derived from (2.5) for the pseudo-integrals.

Theorem 4.1. (Pseudo Minkowski's inequality, decreasing case). Let $f, h : [0,1] \to [0,1]$ be continuous and strictly decreasing functions and μ be the Lebesgue measure on $\mathbb{R}$. If the pseudo-operations are defined by a continuous and decreasing $g : [0,1] \to [0,\infty]$ and functions satisfying

$$0 < m \leq \frac{f(x)}{h(x)} \leq M, \quad \forall x \in [0,1]$$

then the inequality

$$\Big(\int_{[0,1]}^{\oplus} f(x)^s d\mu\Big)^{\frac{1}{s}} + \Big(\int_{[0,1]}^{\oplus} h(x)^s d\mu\Big)^{\frac{1}{s}} \leq 2c\Big(\int_{[0,1]}^{\oplus} (f(x)+h(x))^s d\mu\Big)^{\frac{1}{s}}, \tag{4.1}$$

holds, where $1 \leq s < \infty$, and $c = \dfrac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

Proof. Since $\frac{f(x)}{g(x)} \leq M, f(x) \leq M(f(x)+g(x)) - Mf(x)$. Therefore

$$(M+1)^s f(x)^s \leq M^s (f(x)+g(x))^s$$

and so,

$$f(x)^s \leq \frac{M^s}{(M+1)^s}(f(x)+g(x))^s.$$

Now from Lemma 2.8 (2),

$$\Big(\int_{[0,1]}^{\oplus} f(x)^s d\mu\Big)^{\frac{1}{s}} \leq \Big(\int_{[0,1]}^{\oplus} (\frac{M}{M+1})^s (f(x)+g(x))^s d\mu\Big)^{\frac{1}{s}}. \tag{4.2}$$

Since $\dfrac{M}{M+1} < 1$, from Lemma 2.8 (2), we have

$$\left(\int_{[0,1]}^{\oplus} f(x)^s d\mu\right)^{\frac{1}{s}} \leq \left(\int_{[0,1]}^{\oplus} (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}}. \tag{4.3}$$

On the other hand, since $mg(x) \leq f(x)$, Hence

$$g(x) \leq \frac{1}{m}(f(x)+g(x)) - \frac{1}{m}g(x).$$

Therefore,

$$(\frac{1}{m}+1)^s g(x)^s \leq (\frac{1}{m})^s (f(x)+g(x))^s.$$

and so, from Lemma 2.8 (2),

$$\left(\int_{[0,1]}^{\oplus} g(x)^s d\mu\right)^{\frac{1}{s}} \leq \left(\int_{[0,1]}^{\oplus} (\frac{1}{m+1})^s (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}}. \tag{4.4}$$

Since $\dfrac{1}{m+1} < 1$, from Lemma 2.8 (2) and the inequality (4.4) we have

$$\left(\int_{[0,1]}^{\oplus} g(x)^s d\mu\right)^{\frac{1}{s}} \leq \left(\int_{[0,1]}^{\oplus} (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}}. \tag{4.5}$$

Now with adding the inequalities (4.3) and (4.5) we have

$$\begin{aligned}\left(\int_{[0,1]}^{\oplus} f(x)^s d\mu\right)^{\frac{1}{s}} + \left(\int_{[0,1]}^{\oplus} g(x)^s d\mu\right)^{\frac{1}{s}} &\leq 2\left(\int_{[0,1]}^{\oplus} (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}} \\ &\leq 2c\left(\int_{[0,1]}^{\oplus} (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}}.\end{aligned}$$

The proof is now complete. Q.E.D.

Example 4.2. Let $f, h : [0,1] \to [0,1]$ be two real valued functions as $f(x) = -x+\frac{1}{2}$, $h(x) = -x+\frac{3}{2}$ and μ be the Lebesgue measure on $\mathbb{R}$. Let $s = 1, g(x) = -x$, A straightforward calculus shows that $0 < \frac{4}{3} \leq \frac{f}{g} \leq 2$. Since

$$\begin{aligned}\text{(i)} \int_{[0,1]}^{\oplus} f(x) d\mu &= g^{-1}\int_0^1 g(f(x)) d\mu \\ &= g^{-1}\int_0^1 -(-x+\frac{1}{2}) d\mu \\ &= g^{-1}\int_0^1 (x-\frac{1}{2}) d\mu \\ &= g^{-1}(\frac{1}{2}x^2 - \frac{1}{2}x|_0^1) \\ &= g^{-1}(0) \\ &= 0,\end{aligned}$$

$$\begin{aligned}
\text{(ii)} \int_{[0,1]}^{\oplus} h(x)d\mu &= g^{-1}\int_0^1 g(h(x)d\mu \\
&= g^{-1}\int_0^1 -(-x+\frac{3}{2})d\mu \\
&= g^{-1}\int_0^1 (x-\frac{3}{2})d\mu \\
&= g^{-1}(\frac{1}{2}x^2-\frac{3}{2}x|_0^1) \\
&= g^{-1}(-1) \\
&= 1,
\end{aligned}$$

and

$$\begin{aligned}
\text{(iii)} \int_{[0,1]}^{\oplus} ((f+h)(x))d\mu &= g^{-1}\int_0^1 g((f+h)(x))d\mu \\
&= g^{-1}\int_0^1 g(-2x+2)d\mu \\
&= \\
&= g^{-1}\int_0^1 (2x-2)d\mu \\
&= g^{-1}(x^2-2x|_0^1) \\
&= g^{-1}(-1) \\
&= 1.
\end{aligned}$$

Therefore

$$\begin{aligned}
1=0+1=\Big(\int_{[0,1]}^{\oplus} fd\mu\Big)+\Big(\int_{[0,1]}^{\oplus} gd\mu\Big) &\leq 2c\Big(\int_{[0,1]}^{\oplus}(f+g)d\mu\Big) \\
&\leq 2\times c\times 1 \\
&= 2c.
\end{aligned}$$

Theorem 4.3. (Pseudo Minkowski inequality, increasing case). Let $f, h : [0,1] \to [0,1]$ be continuous and strictly increasing functions and μ be the Lebesgue measure on $\mathbb{R}$. If the pseudo-operations are defined by a continuous and increasing $g : [0,1] \to [0,1]$ and functions satisfying

$$0 < m \leq \frac{f(x)}{h(x)} \leq M, \quad \forall x \in [0,1]$$

then the inequality

$$\left(\int_{[0,1]}^{\oplus} f(x)^s d\mu\right)^{\frac{1}{s}} + \left(\int_{[0,1]}^{\oplus} h(x)^s d\mu\right)^{\frac{1}{s}} \leq nc\left(\int_{[0,1]}^{\oplus} (f(x)+h(x))^s d\mu\right)^{\frac{1}{s}}, \tag{4.6}$$

holds, where $1 \leq s < \infty$ and $c = \dfrac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

Proof. By using the same argument in Theorem 4.1 proof is obvious. Q.E.D.

Now we generaliz the Minkowski type inequality by the semiring $([0,1], \max, \odot)$, where $\odot$ is generated.

Theorem 4.4. Let $f, h : [0,1] \to [0,1]$ be continuous and strictly decreasing functions and let m be the same as in Theorem 2.6. If $\odot$ is represented by an decreasing multiplicative generator g and functions satisfying

$$0 < m \leq \frac{f(x)}{h(x)} \leq M, \quad \forall x \in [0,1]$$

then the inequality

$$\left(\int_{[0,1]}^{\sup} f^s \odot dm\right)^{\frac{1}{s}} + \left(\int_{[0,1]}^{\sup} h^s \odot dm\right)^{\frac{1}{s}} \leq nc \left(\int_{[0,1]}^{\sup} (f+h)^s \odot dm\right)^{\frac{1}{s}}, \tag{4.7}$$

holds, where $1 \leq s < \infty, n \geq 2$ and $c = \dfrac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

Proof. Since $\frac{f(x)}{g(x)} \leq M, f \leq M\left(f(x)+g(x)\right) - Mf(x)$. Therefore

$$(M+1)^s f(x)^s \leq M^s (f(x)+g(x))^s$$

and so,

$$f(x)^s \leq \frac{M^s}{(M+1)^s}(f(x)+g(x))^s.$$

Now,

$$\left(\int_{[0,1]}^{\oplus_\lambda} f(x)^s \odot dm\right)^{\frac{1}{s}} \leq \left(\int_{[0,1]}^{\oplus_\lambda} (\frac{M}{M+1})^s (f(x)+g(x))^s \odot dm\right)^{\frac{1}{s}}.$$

Since $\dfrac{M}{M+1} < 1$, so

$$\left(\int_{[0,1]}^{\oplus_\lambda} f(x)^s \odot dm\right)^{\frac{1}{s}} \leq \left(\int_{[0,1]}^{\oplus_\lambda} (f(x)+g(x))^s \odot dm\right)^{\frac{1}{s}}.$$

It follows that

$$\left(\lim_{\lambda\to\infty} \int_{[0,1]}^{\oplus_\lambda} f(x)^s \odot dm\right)^{\frac{1}{s}} \leq \left(\lim_{\lambda\to\infty} \int_{[0,1]}^{\oplus_\lambda} (f(x)+g(x))^s \odot dm\right)^{\frac{1}{s}}.$$

Finally,

$$\left(\int_{[0,1]}^{\sup} f(x)^s \odot dm\right)^{\frac{1}{s}} \leq \left(\int_{[0,1]}^{\sup} (f(x)+g(x))^s \odot dm\right)^{\frac{1}{s}}. \tag{4.8}$$

On the other hand, since $mg(x) \leq f(x)$, hence

$$g(x) \leq \frac{1}{m}(f(x)+g(x)) - \frac{1}{m}g(x).$$

Therefore,

$$(\frac{1}{m}+1)^s g(x)^s \leq (\frac{1}{m})^s (f(x)+g(x))^s$$

and so,

$$\left(\int_{[0,1]}^{\oplus_\lambda} g(x)^s \odot dm\right)^{\frac{1}{s}} \leq \left(\int_{[0,1]}^{\oplus_\lambda} (\frac{1}{m+1})^s (f(x)+g(x))^s \odot dm\right)^{\frac{1}{s}}.$$

Since $\frac{1}{m+1} < 1$, so

$$\left(\int_{[0,1]}^{\oplus_\lambda} g(x)^s \odot dm\right)^{\frac{1}{s}} \leq \left(\int_{[0,1]}^{\oplus_\lambda} (f(x)+g(x))^s \odot dm\right)^{\frac{1}{s}}.$$

It follows that

$$\left(\lim_{\lambda\to\infty}\int_{[0,1]}^{\oplus_\lambda} g(x)^s \odot dm\right)^{\frac{1}{s}} \leq \left(\lim_{\lambda\to\infty}\int_{[0,1]}^{\oplus_\lambda} (f(x)+g(x))^s \odot dm\right)^{\frac{1}{s}}.$$

Finally,

$$\left(\int_{[0,1]}^{\sup} g(x)^s \odot dm\right)^{\frac{1}{s}} \leq \left(\int_{[0,1]}^{\sup} (f(x)+g(x))^s \odot dm\right)^{\frac{1}{s}}. \tag{4.9}$$

Now with adding the inequalities (4.8) and (4.9):

$$\begin{aligned}\left(\int_{[0,1]}^{\sup} f(x)^s \odot dm\right)^{\frac{1}{s}} + \left(\int_{[0,1]}^{\sup} g(x)^s \odot dm\right)^{\frac{1}{s}} &\leq 2\left(\int_{[0,1]}^{\sup} (f(x)+g(x))^s \odot dm\right)^{\frac{1}{s}} \\ &\leq 2c\left(\int_{[0,1]}^{\sup} (f(x)+g(x))^s \odot dm\right)^{\frac{1}{s}}.\end{aligned}$$

The proof is now complete. Q.E.D.

Example 4.5. Let $f, h : [0,1] \to [0,\infty)$ be a μ-measurable, and $g^\lambda(x) = x^{-\lambda}$. So

$$x \oplus y = (x^{-\lambda} + y^{-\lambda})^{-\lambda} \quad \text{and} \quad x \odot y = xy.$$

Therefore Relation (4.7) reduces on the following inequality:

$$\sup\left((f(x)^s)^{\frac{1}{s}} + \psi(x)\right) + \sup\left((h(x)^s)^{\frac{1}{s}} + \psi(x)\right) \leq nc\sup\left((f+h)^s(x) + \psi(x)\right).$$

where ψ is from Theorem 2.6.

Theorem 4.6. Let $f, h : [0,1] \to [0,\infty)$ are continuous and strictly increasing functions and let m be the same as in theorem 2.6. If $\odot$ is represented by an increasing multiplicative generator g and functions satisfying

$$0 < m \leq \frac{f}{h} \leq M, \quad \forall x \in [0,1]$$

then the inequality

$$\left(\int_{[0,1]}^{\sup} f^s \odot dm\right)^{\frac{1}{s}} + \left(\int_{[0,1]}^{\sup} h^s \odot dm\right)^{\frac{1}{s}} \leq 2c\left(\int_{[0,1]}^{\sup} (f+h)^s \odot dm\right)^{\frac{1}{s}}, \tag{4.10}$$

holds, where $1 \leq s < \infty$ and $c = \dfrac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

Proof. The proof is similar to Theorem 4.4. Q.E.D.

Note that third important case $\oplus$ =max and $\odot$ =min has been studied in [38] and the Pseudo-integrals in such a case yields the Sugeno integral.

Conclusion: The classical Minkowski inequality is an important result in theoretical and applied fields. This paper proposed a Minkowski type inequality for fuzzy antegrals. Also, we proved this inequality for pseudo integrals: The first class is including the pseudo-integral based on a function reduces on the $g-$integral, where pseudo-addition and pseudo-multiplication are defined by a monotone and continuous function g. The second class is including the pseudo-integral based on the semiring $([a,b], \max, \odot)$ is given by $\sup-$measure, where $x \odot y$ is generated by $g^{-1}(g(x)g(y))$.

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